Hyperbolic Tilings and their Orbits

Yotam Smilansky, Rutgers

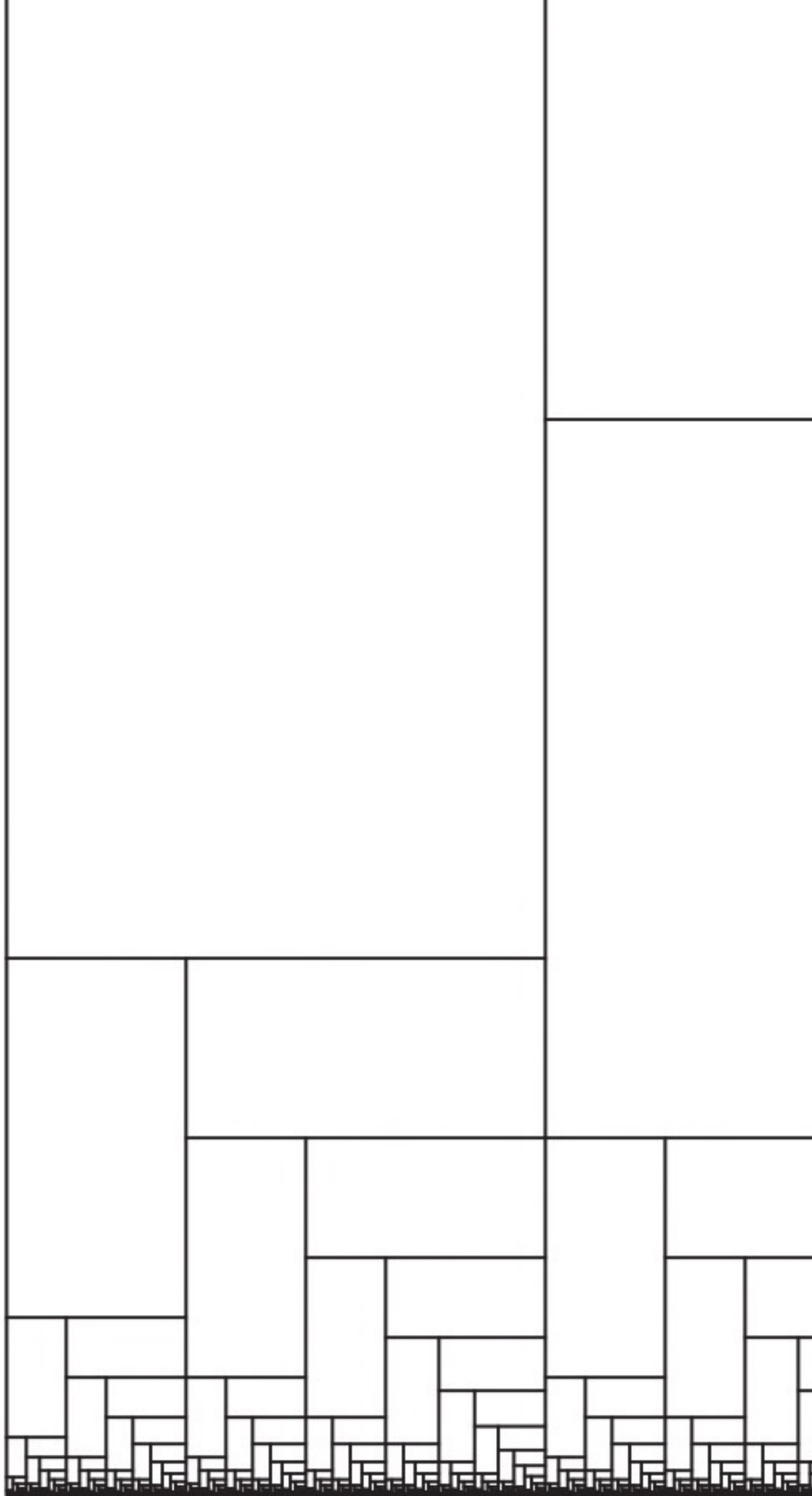
Action Now Wandering Seminar

Tel Aviv University, 2022

Based on joint work with Yaar Solomon

Plan of Talk

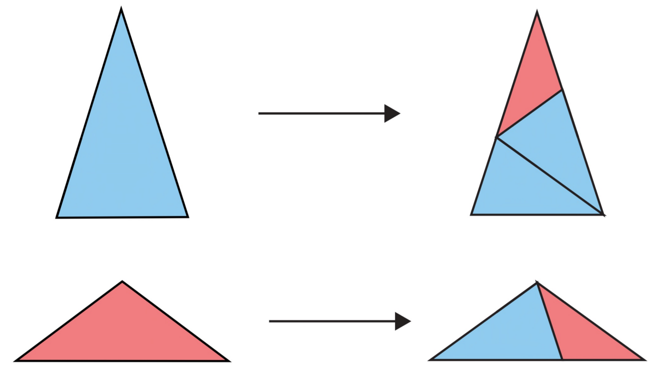
- Substitution Schemes and Tilings
- Hyperbolic Tilings
- Statistics and Orbits



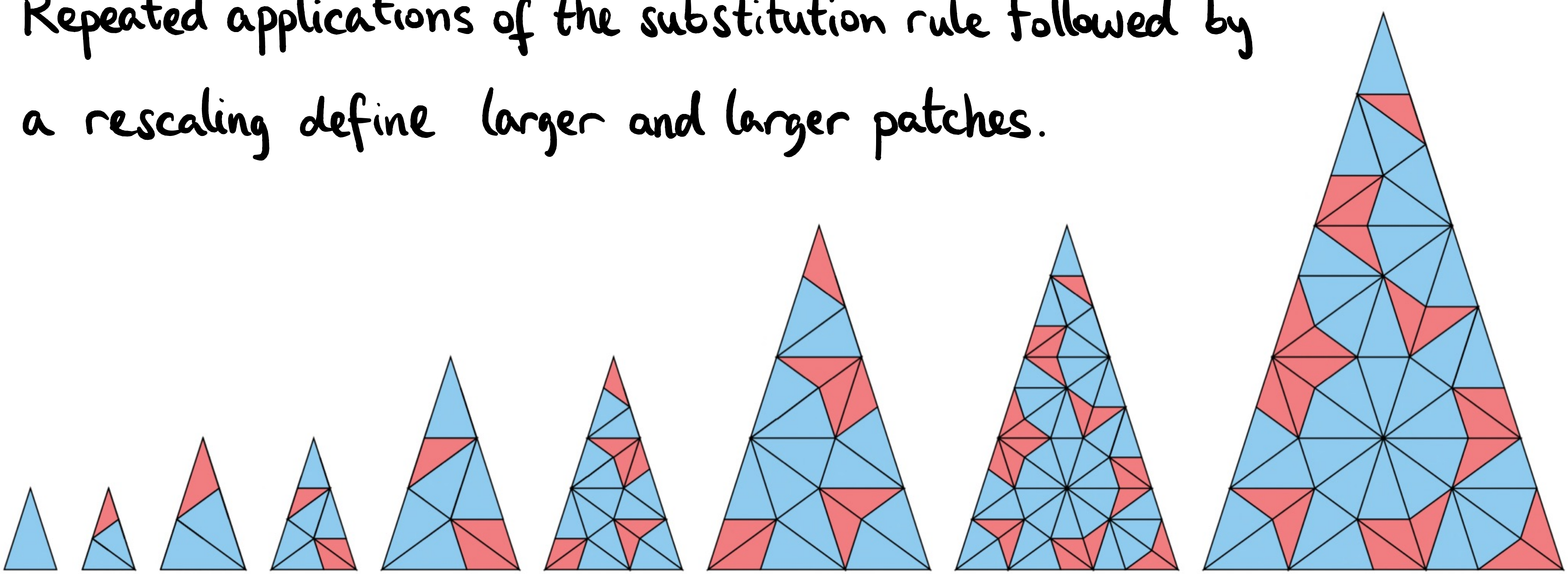
Substitution Tilings in $\mathbb{R}^d$

A tiling is a collection of tiles with disjoint interiors that covers $\mathbb{R}^d$.

A substitution rule on a set of prototiles is a tessellation of each prototile by rescaled prototiles, with a fixed scale $\in (0,1)$

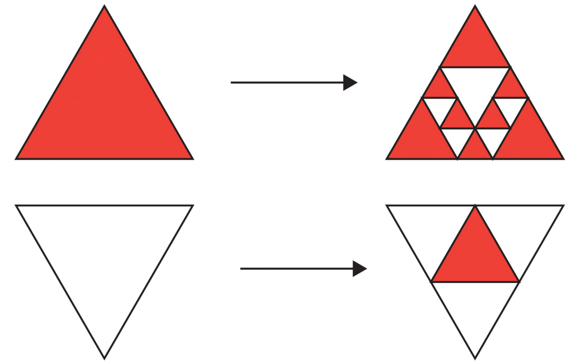


Repeated applications of the substitution rule followed by a rescaling define larger and larger patches.

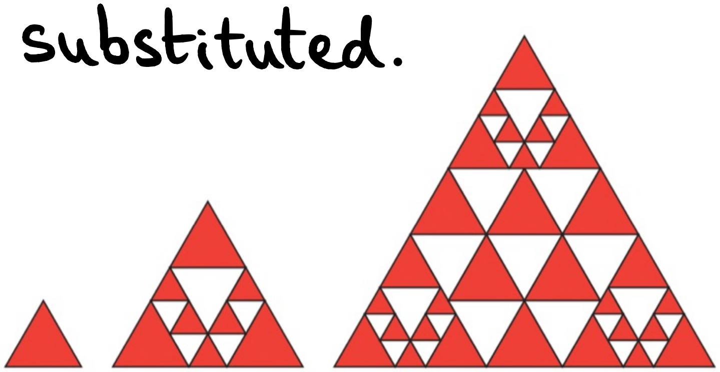


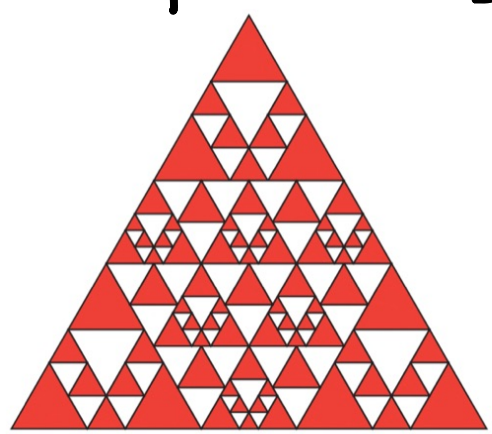
Multiscale Substitution Tilings in $\mathbb{R}^d$ [SS '21]

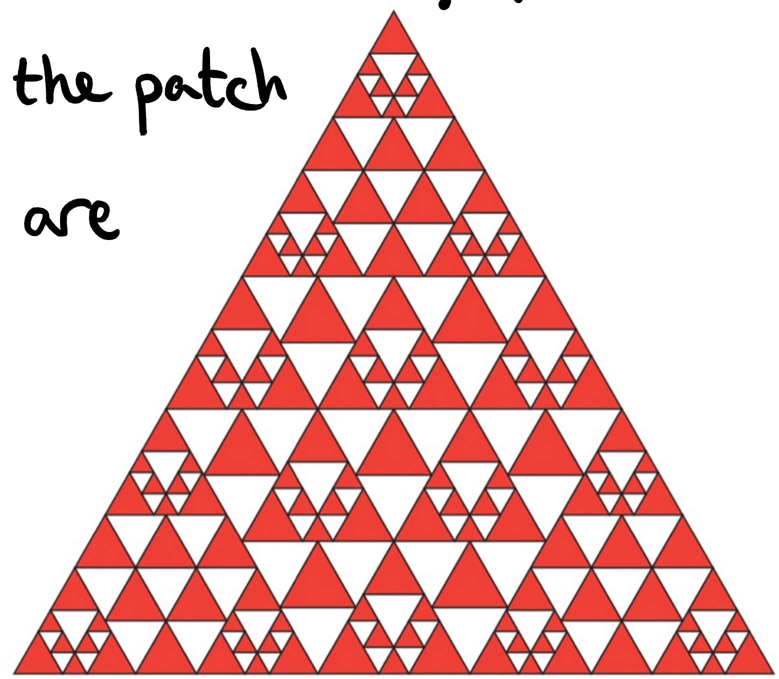
A multiscale substitution scheme σ in $\mathbb{R}^d$ consists of a substitution rule on unit volume prototiles $T_1, \ldots, T_n$, where various different scales appear and satisfy a simple incommensurabily condition.

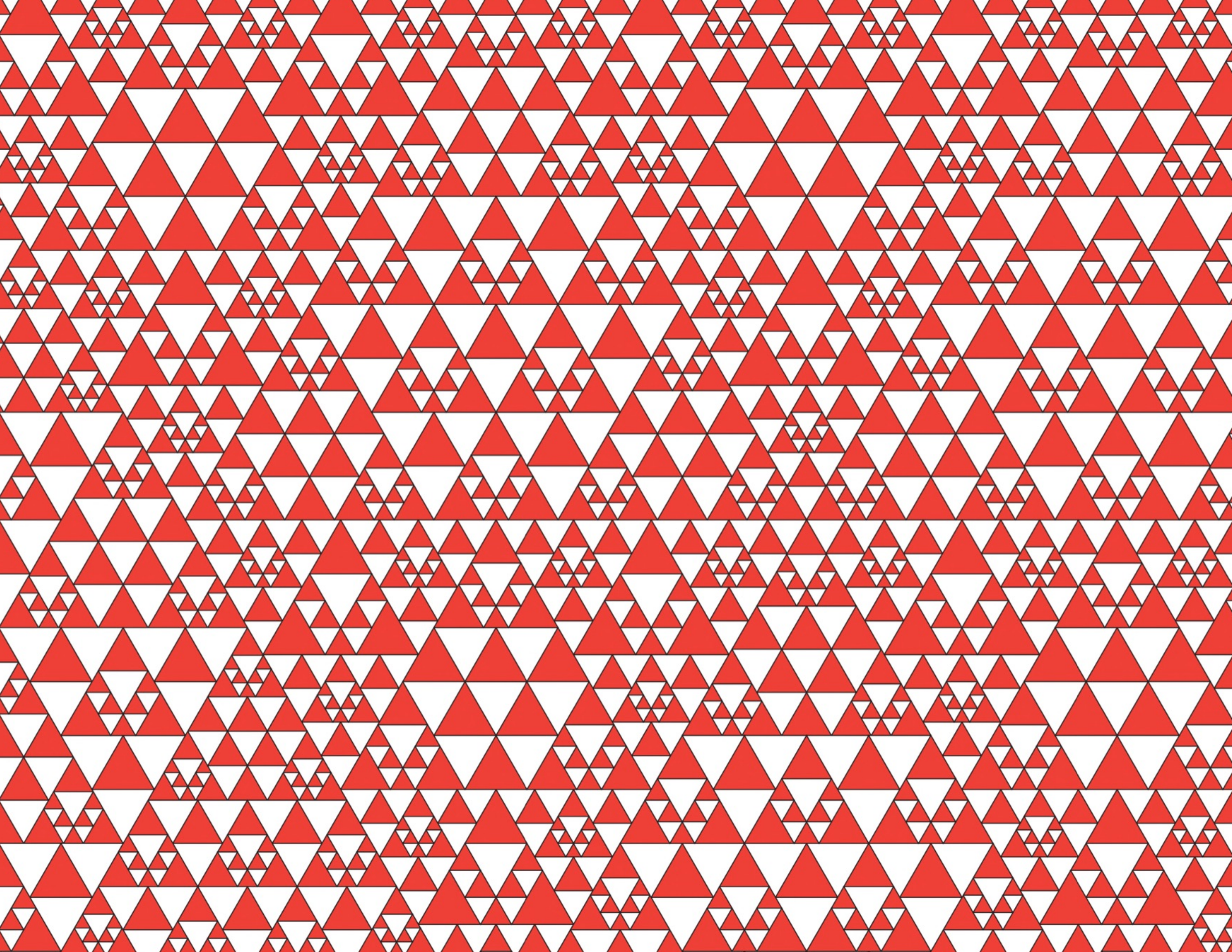


A time-dependent substitution semiflow F_t defines a family of patches: At time $t=0$ $F_t(T)=T$, and as t increases the patch is inflated by e^t and tiles of volume >1 are substituted.







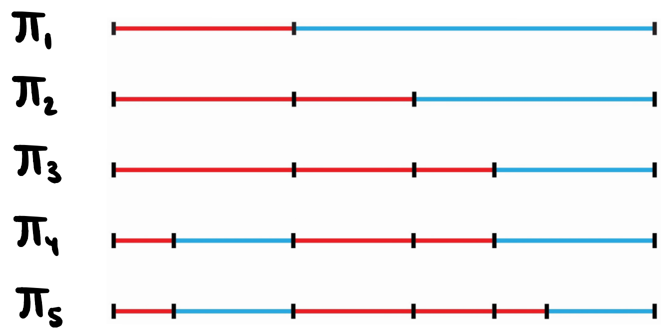


Useful Example: α-Kakutani Schemes

- substitution scheme

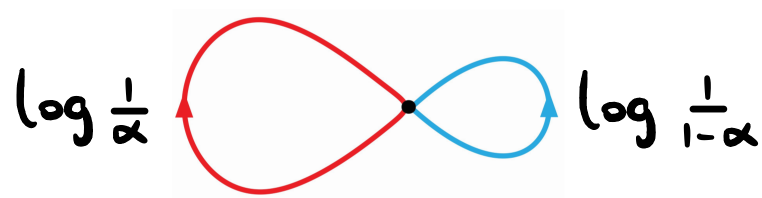
- Kakutani sequence of partitions π_m [Kakutani '76, S '20]

always split longest interval



- tilings of $\mathbb{R}$ [SS '21]

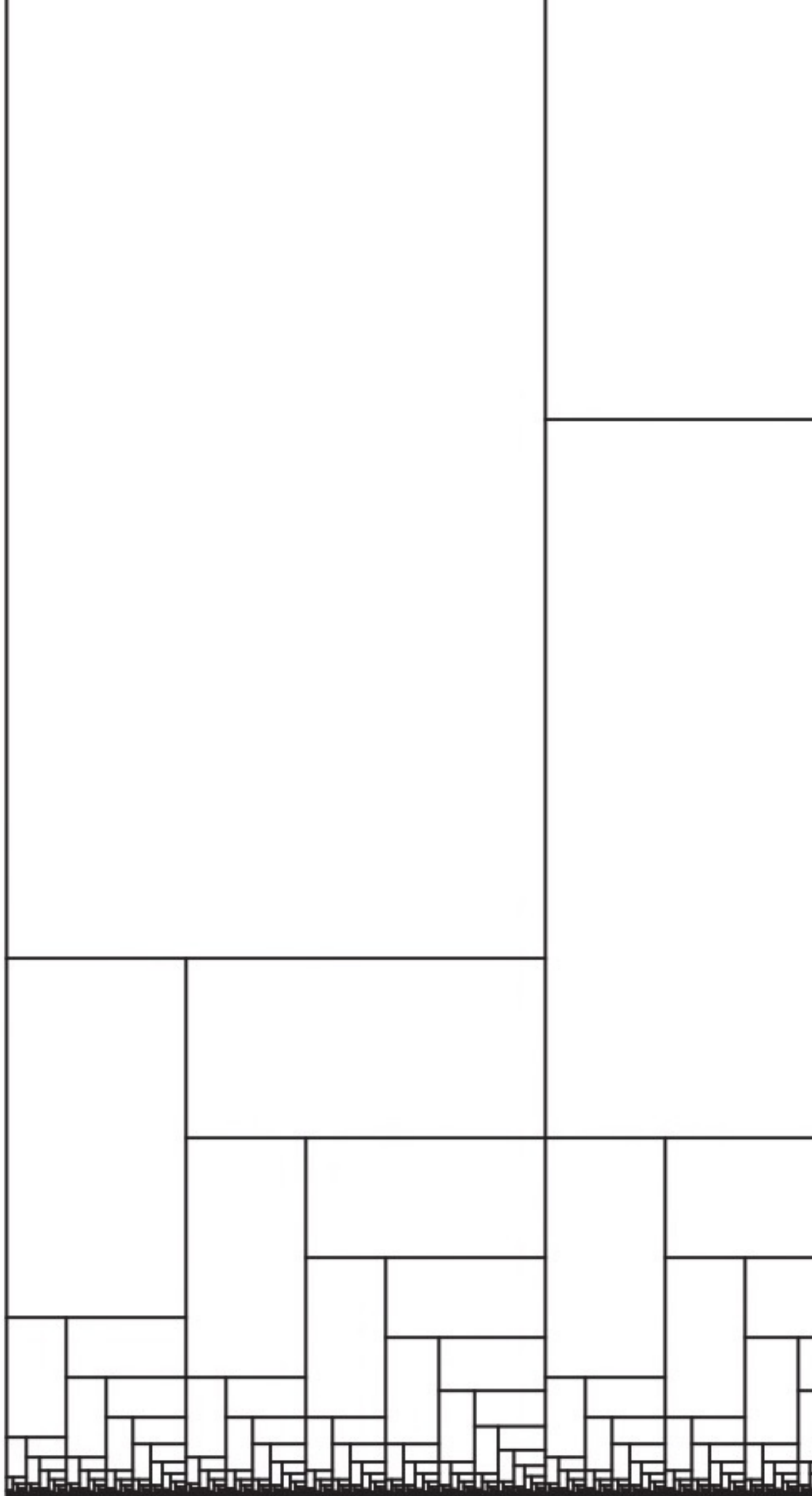
- associated graph G_σ [KSS '20]



σ is incommensurable if G_σ contains two closed paths of lengths $\frac{a}{b} \notin \mathbb{Q}$.

Plan of Talk

- Substitution Schemes and Tilings
- Hyperbolic Tilings
- Statistics and Orbits



From Schemes in $\mathbb{R}^d$ to Tilings of $\mathbb{H}^{d+1}$

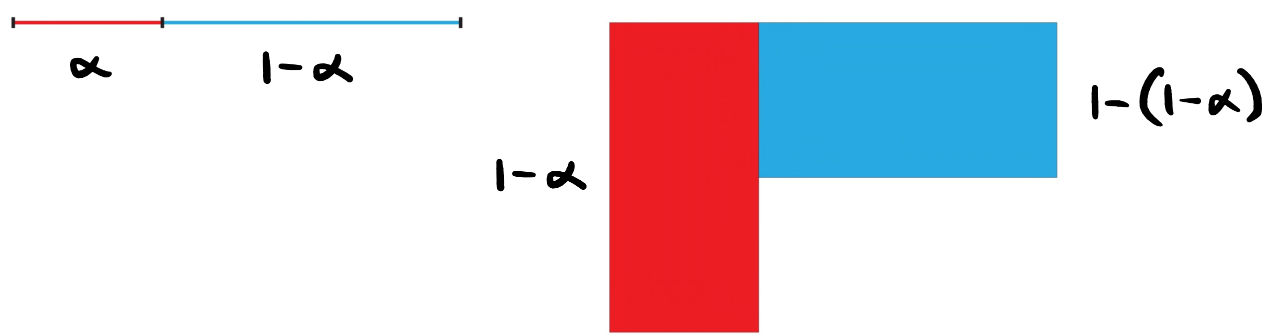
- upper half-space $\quad \mathbb{H}^{d+1} := \{(x,s) : x=(x_1,\dots,x_d)\in\mathbb{R}^d,\ s>0\}$

Two continuous actions by hyperbolic isometries:

- horospheric $\mathbb{R}^d$-action $\quad h_y(x,s) = (y+x, s) \quad$ for $y\in\mathbb{R}^d$
- geodesic $\mathbb{R}$-action $\quad g_t(x,s) = (e^t x, e^t s) \quad$ for $t\in\mathbb{R}$

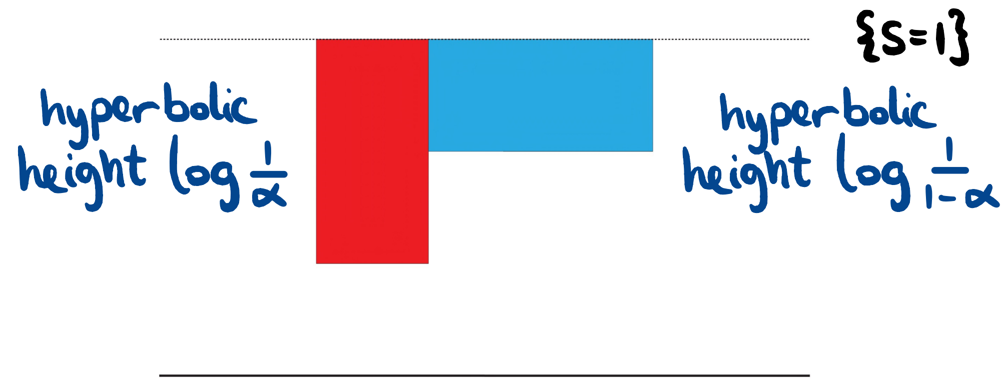
satisfying $\quad g_t\circ h_y = h_{e^t y}\circ g_t$

Basic idea (Petite): give every Euclidean tile a height corresponding to its scale, and glue tiles together instead of substituting



Gluing Procedure

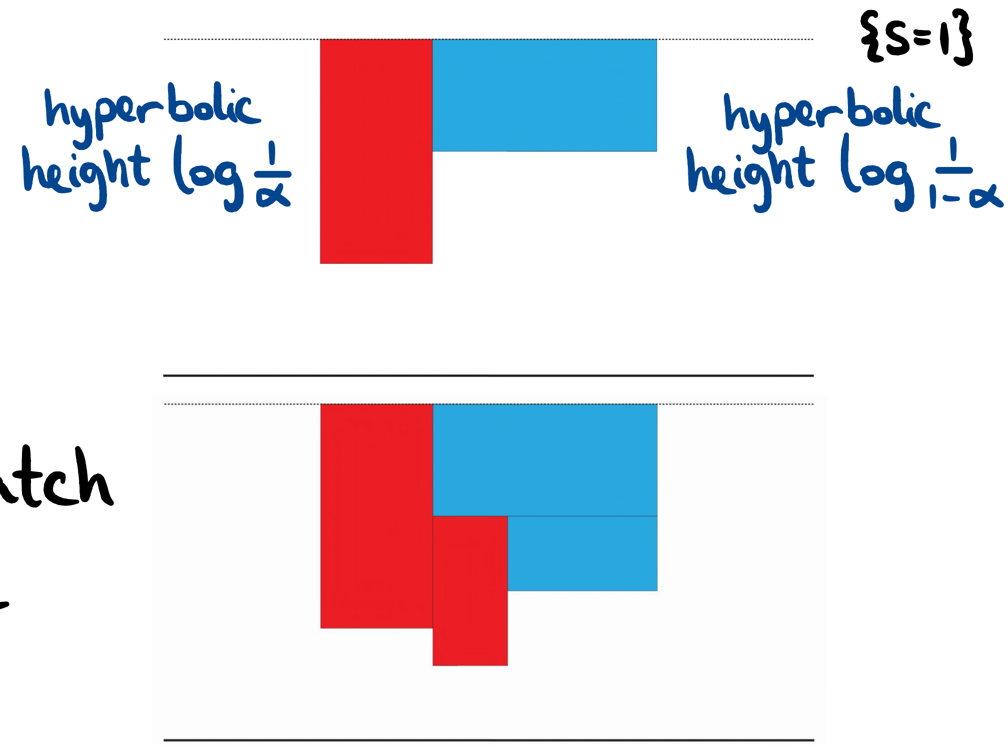
- Position the patch in $\mathbb{H}^{d+1}$ aligned to the horosphere $\{s=1\}$



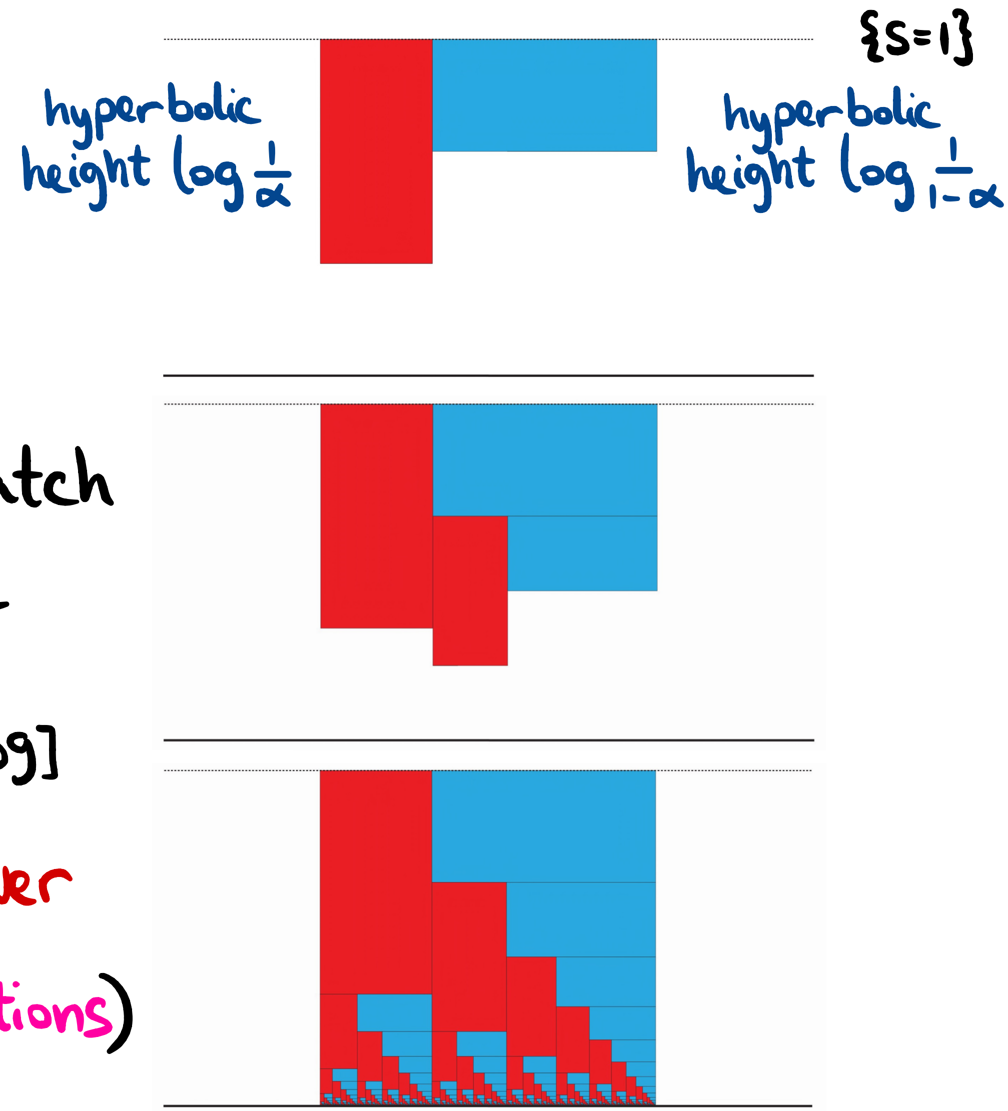
Gluing Procedure

- Position the patch in $\mathbb{H}^{d+1}$ aligned to the horosphere $\{s=1\}$

- Glue an isometric copy of the patch to the bottom of a tile, repeat



Gluing Procedure

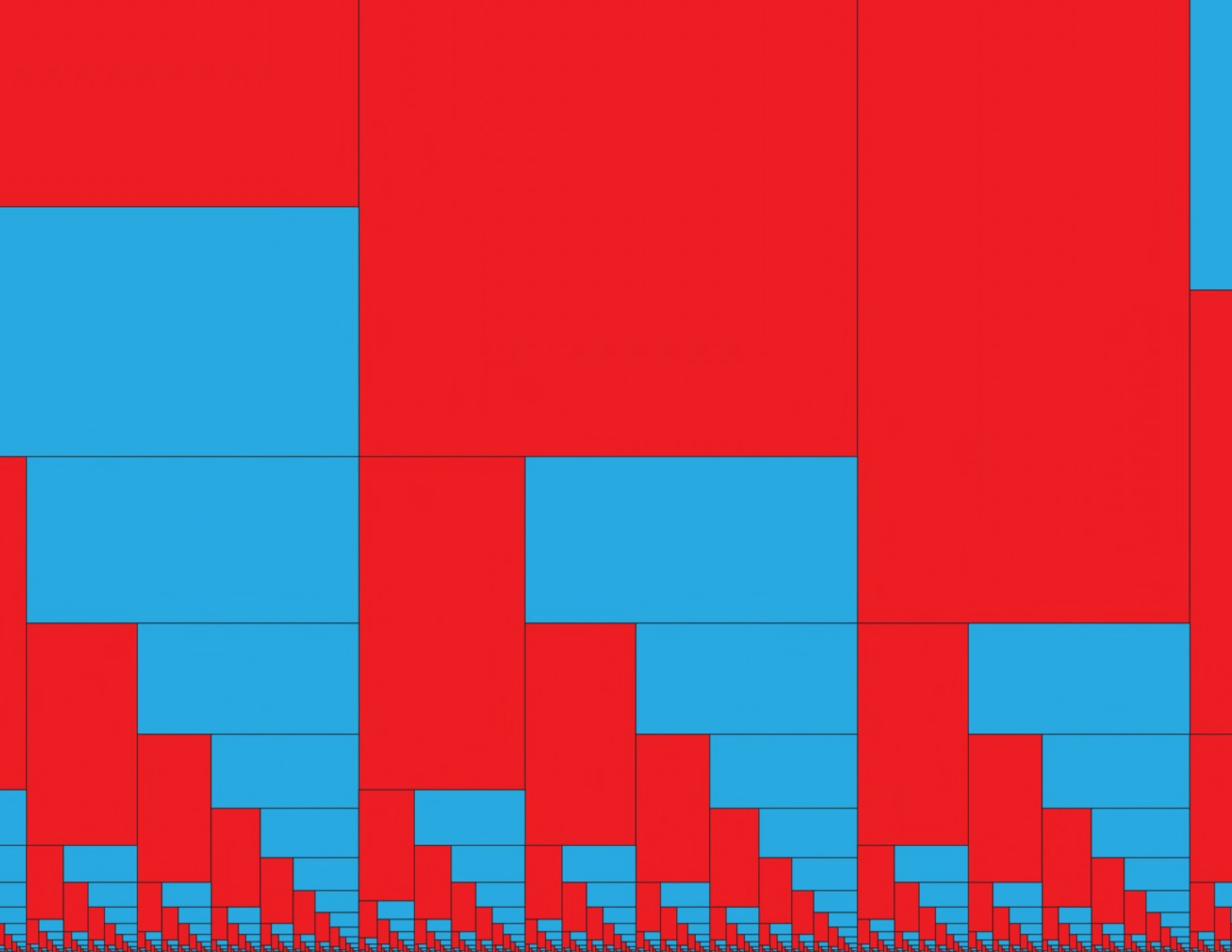


- Position the patch in $\mathbb{H}^{d+1}$ aligned to the horosphere $\{s=1\}$
- Glue an isometric copy of the patch to the bottom of a tile, repeat

  [after Dolbilin, Frettlöh '09]
- The limit object is called a tower (suspended normalized Kakutani partitions)
- Tilings of $\mathbb{H}^{d+1}$ are partial limits of towers under g_t as $t \to \infty$.

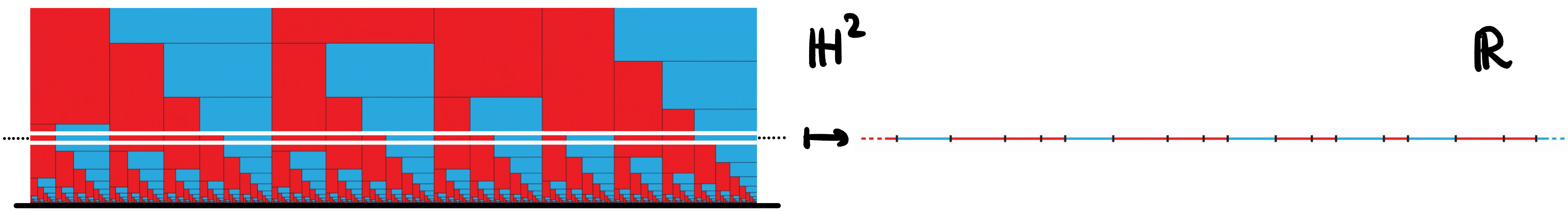
  The tiling space X_σ^{hyp} is the collection of all such limits,

  X_σ^{hyp} is invariant under both h_y and g_t.



Liftings of Multiscale Substitution Tilings

Theorem Let $T \in X_\sigma^{hyp}$ be a $d+1$-dimensional hyperbolic tiling, then the map $T \mapsto T \cap \{s=1\}$ defines a d-dimensional multiscale substitution tiling generated by σ.



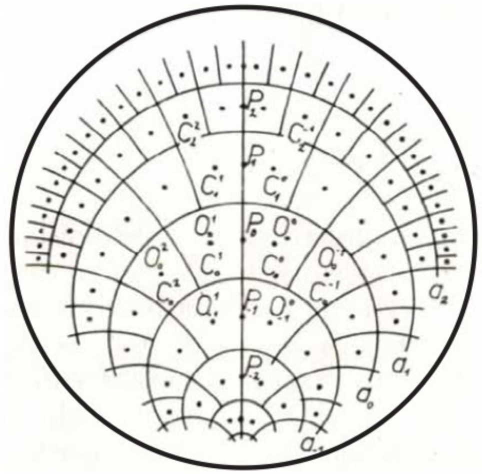
Finitely many tiles up to hyperbolic isometries

Infinitely many tiles up to Euclidean isometries (assuming incommensurability)

- horospheric $\mathbb{R}^d$-action h_y $\rightarrow$ translation action in $\mathbb{R}^d$ "space"
- geodesic $\mathbb{R}$-action g_t $\rightarrow$ substitution semiflow F_t "time"

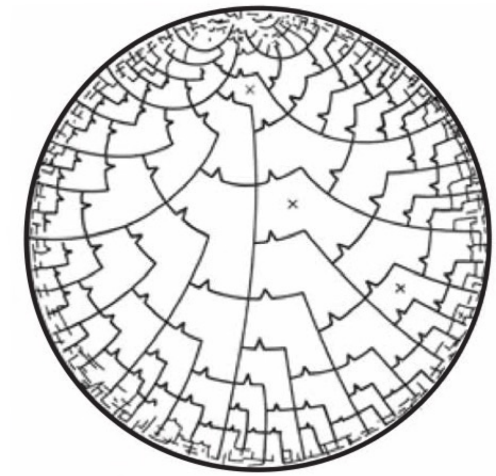
Related Hyperbolic Constructions

Poincaré disk model



$\alpha = \frac{1}{2}$

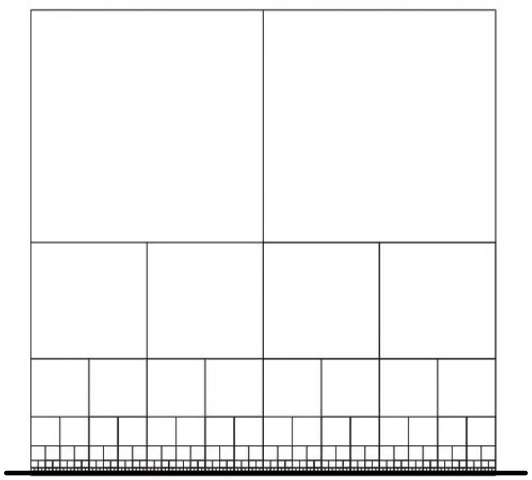
[Böröczky '74]



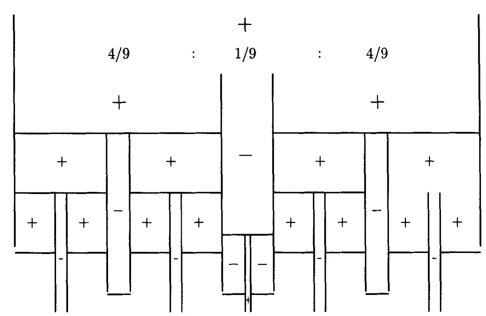
$\alpha = 1 - \frac{1}{\varphi}$

[Penrose '79]

Upper half-plane model



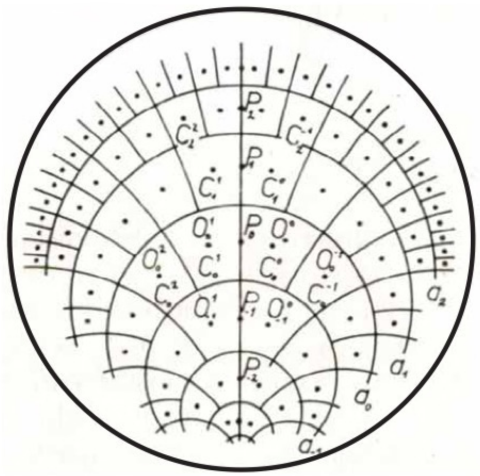
Binary tiling



Numeration systems
[Kamae '05]

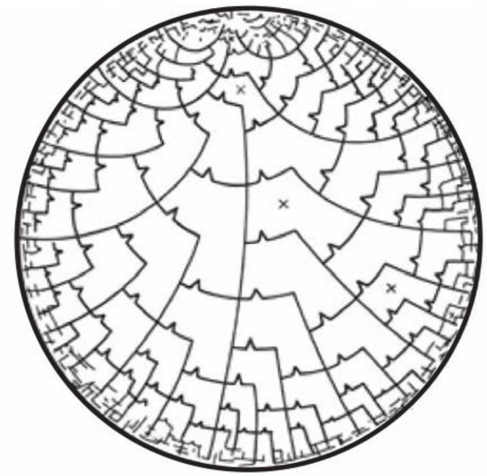
Related Hyperbolic Constructions

Poincaré disk model



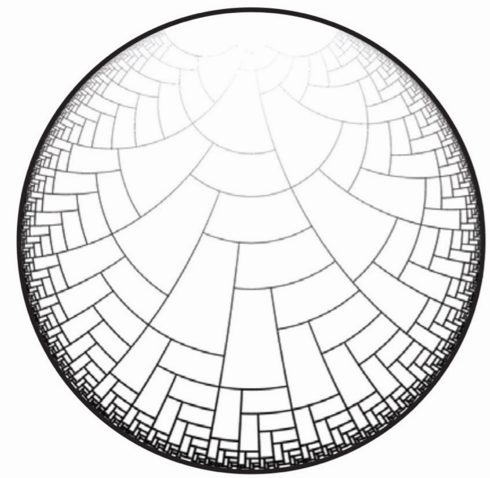
$\alpha = \frac{1}{2}$

[Böröczky '74]



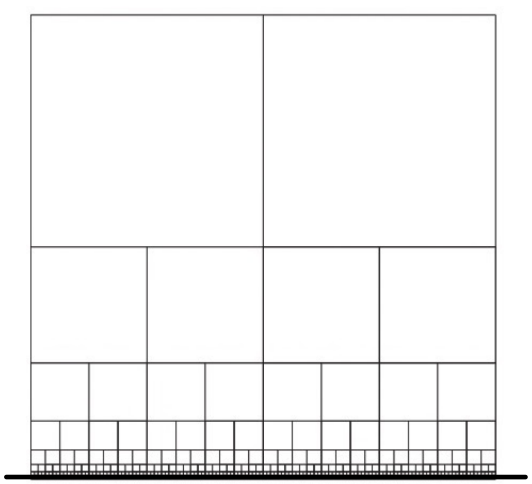
$\alpha = 1 - \frac{1}{\varphi}$

[Penrose '79]

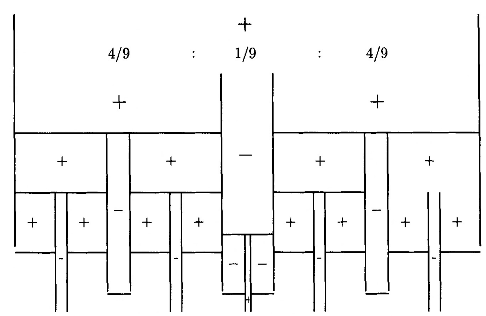


$\alpha = \frac{1}{3}$

Upper half-plane model

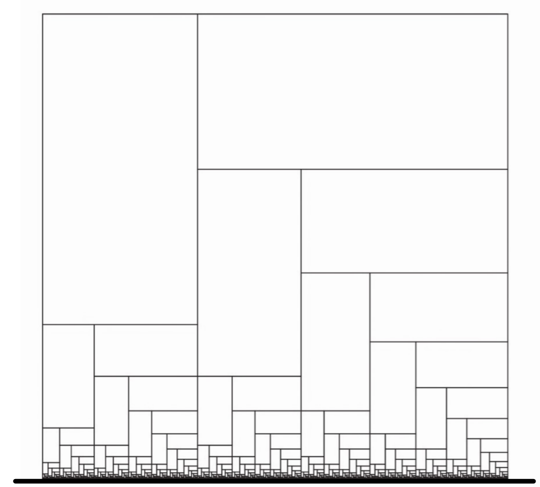


Binary tiling



Numeration systems

[Kamae '05]



$\alpha = \frac{1}{3}$

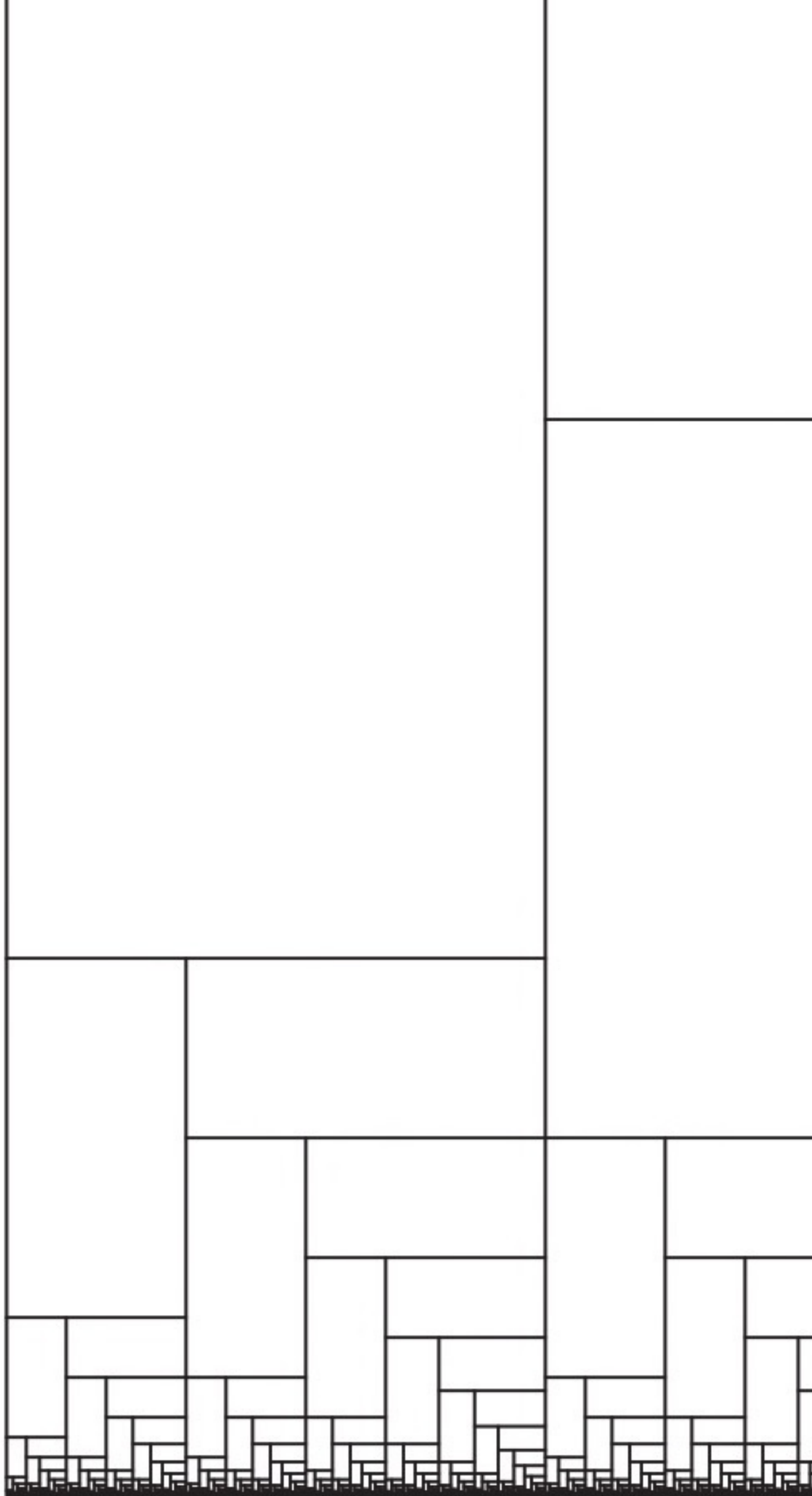
Related Hyperbolic Constructions

Escher's Regular Division of The Plane VI '57

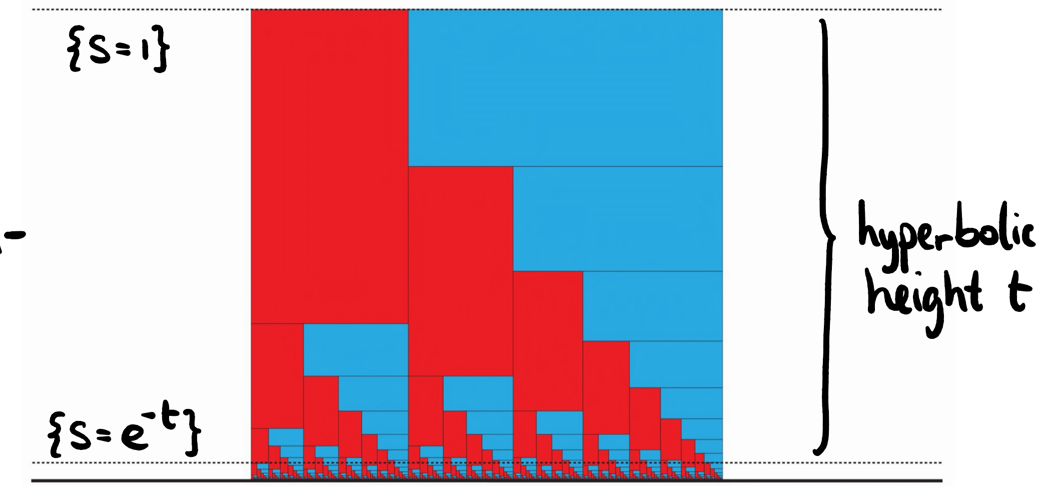
$\alpha = \frac{1}{2}$

Plan of Talk

- Substitution Schemes and Tilings
- Hyperbolic Tilings
- Statistics and Orbits



Counting in Towers



Theorem Let σ be an incommensurable scheme in $\mathbb{R}^d$. Then:

$$\#\left\{\begin{array}{l}\text{tiles of type } j \text{ above} \\ \{s=e^{-t}\} \text{ inside a tower}\end{array}\right\} \sim \frac{[v^T 1]_j}{v^T H_\sigma 1} \cdot e^{dt}, \quad t \to \infty$$

(Paths in G_σ of length $\leq t$)

$$\#\left\{\begin{array}{l}\text{tiles of type } j \text{ intersecting} \\ \{s=e^{-t}\} \text{ inside a tower}\end{array}\right\} \sim \frac{[v^T (S_\sigma - V_\sigma) 1]_j}{v^T H_\sigma 1} \cdot e^{dt}, \quad t \to \infty$$

(Walks in G_σ of length $= t$)

combinatorics matrix $\displaystyle (S_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} 1$

entropy matrix $\displaystyle (H_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} -\mathrm{vol}(T) \cdot \log \mathrm{vol}(T)$

volume matrix $\displaystyle (V_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} \mathrm{vol}(T)$

v^T = left Perron-Frobenius eigenvector of V_σ

The Horospheric Flow

horospheric $\mathbb{R}^d$-action: $h_y(x,s) = (y+x,s)$ for $y \in \mathbb{R}^d$

Let σ be an incommensurable scheme in $\mathbb{R}^d$. Then

Theorem The dynamical system (X_σ^{hyp}, h_y) is minimal, that is, every orbit is dense.

Theorem Tilings in X_σ^{hyp} have no horospheric periods, that is, if $T \in X_\sigma^{hyp}$ and $y \in \mathbb{R}^d$ satisfy $h_y(T) = T$ then $y = 0$.

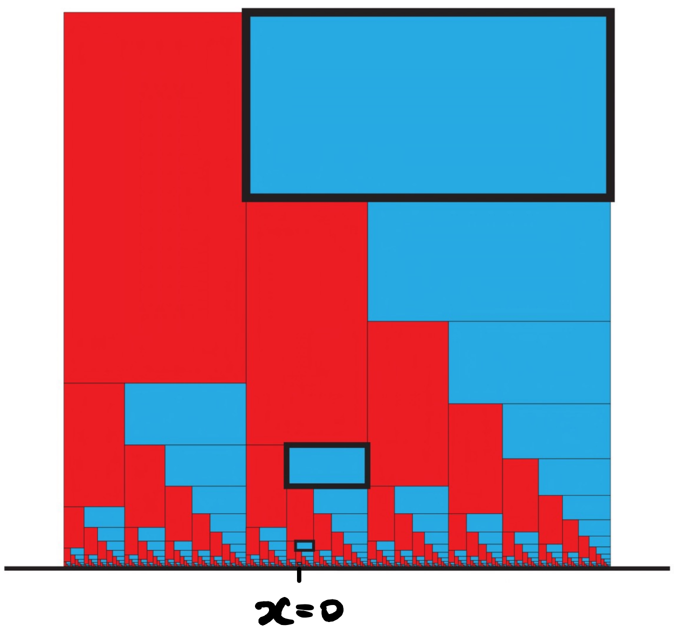
Indeed, then $g_t(T) = g_t(h_y(T)) = h_{e^t y}(g_t(T)) \in X_\sigma^{hyp}$ has period $e^t y$ but is also arbitrarily close to a translation of T with period y, which is impossible.

The Geodesic Flow

geodesic $\mathbb{R}$-action: $g_t(x,s) = (e^t x, e^t s)$ for $t \in \mathbb{R}$

Let σ be an incommensurable scheme in $\mathbb{R}^d$. Then

Theorem The dynamical system (X_σ^{hyp}, g_t) has dense orbits, periodic orbits (and orbits that are neither).



The tiles that intersect the s-axis define an infinite word in the tile alphabet:

periodic words $\Longrightarrow$ periodic g_t orbits

words that contain every finite legal word $\Longrightarrow$ dense g_t orbits

The Geodesic Flow

Theorem (Prime orbit theorem following [Parry, Pollicott '83])

$$\pi_\sigma(t) = \#\{\text{periodic orbits } \tau \text{ with minimal period } \lambda(\tau) \leq t\} \sim \frac{e^{dt}}{dt}, \quad t \to \infty$$

- The tiling zeta function $\zeta_\sigma(s) := \prod_\tau \left(1 - e^{-\lambda(\tau)s}\right)^{-1}$
- $\zeta_\sigma(s) = \dfrac{1}{\det(I - M_\sigma(s))}$, where $(M_\sigma(s))_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} \mathrm{vol}(T)^s$
- $\dfrac{\zeta_\sigma'(s)}{\zeta_\sigma(s)} = -\sum_\tau \sum_{k=1}^\infty \lambda(\tau) e^{-k\lambda(\tau)s} = -s\mathcal{L}\{S(t)\}(s)$, where $S(t) = \sum_{\tau : \lambda(\tau) \leq t} \left\lfloor \frac{t}{\lambda(\tau)} \right\rfloor \lambda(\tau)$
- Laplace transform $\mathcal{L}\{S(t)\}$ has a simple pole at $s = d$ with residue $\frac{1}{d}$
- Wiener-Ikehara: $S(t) \sim \dfrac{e^{dt}}{d}$, $t \to \infty$.
- $S(t) = \sum_{\tau : \lambda(\tau) \leq t} \left\lfloor \frac{t}{\lambda(\tau)} \right\rfloor \lambda(\tau) \leq t \sum_{\tau : \lambda(\tau) \leq t} 1 = t\pi_\sigma(t)$ (upper bound more technical)

Thank You!